Symmetries and Diagram Algebras

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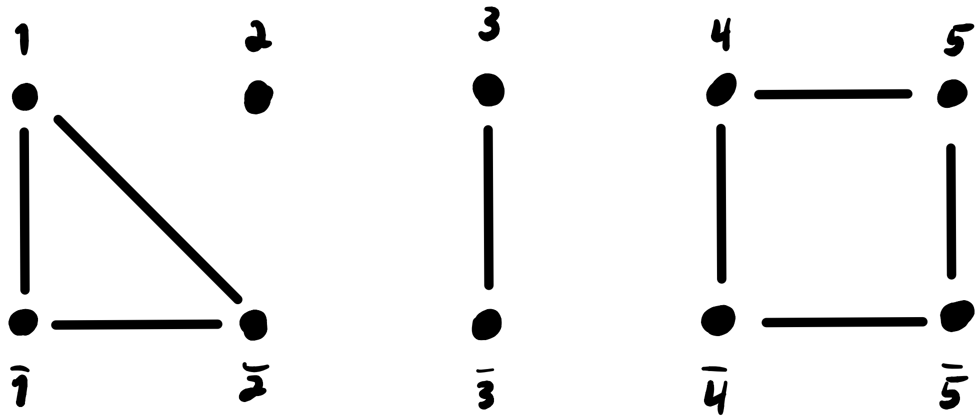
Foreshadowing

A Monoid Structure on Diagrams

An example of what we'll call a partition diagram:



Key features

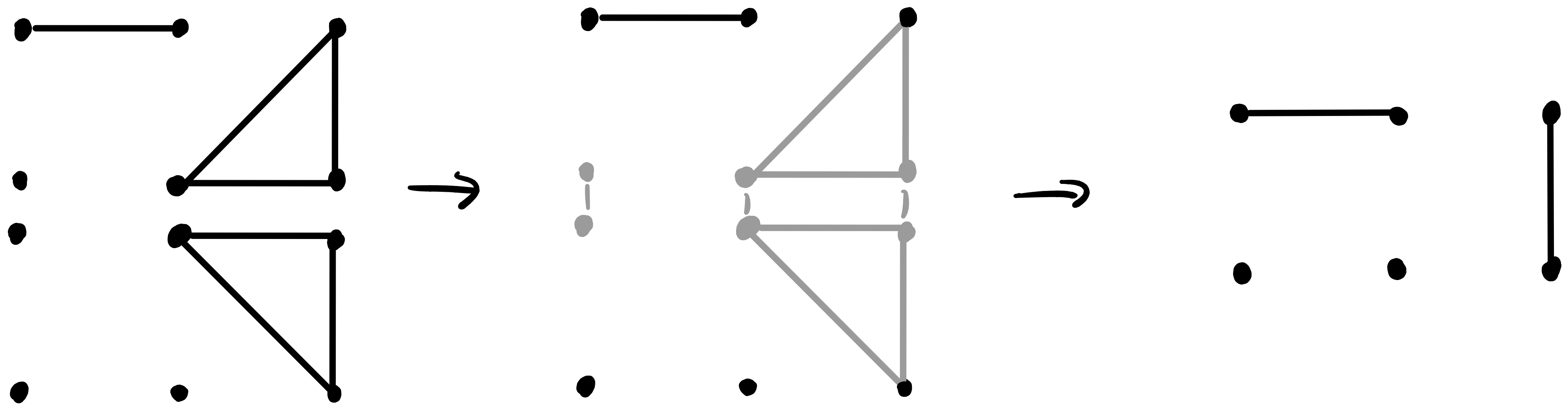
- Has r labeled vertices on top and bottom for some $r > 0$
- The vertices are grouped into connected components by edges

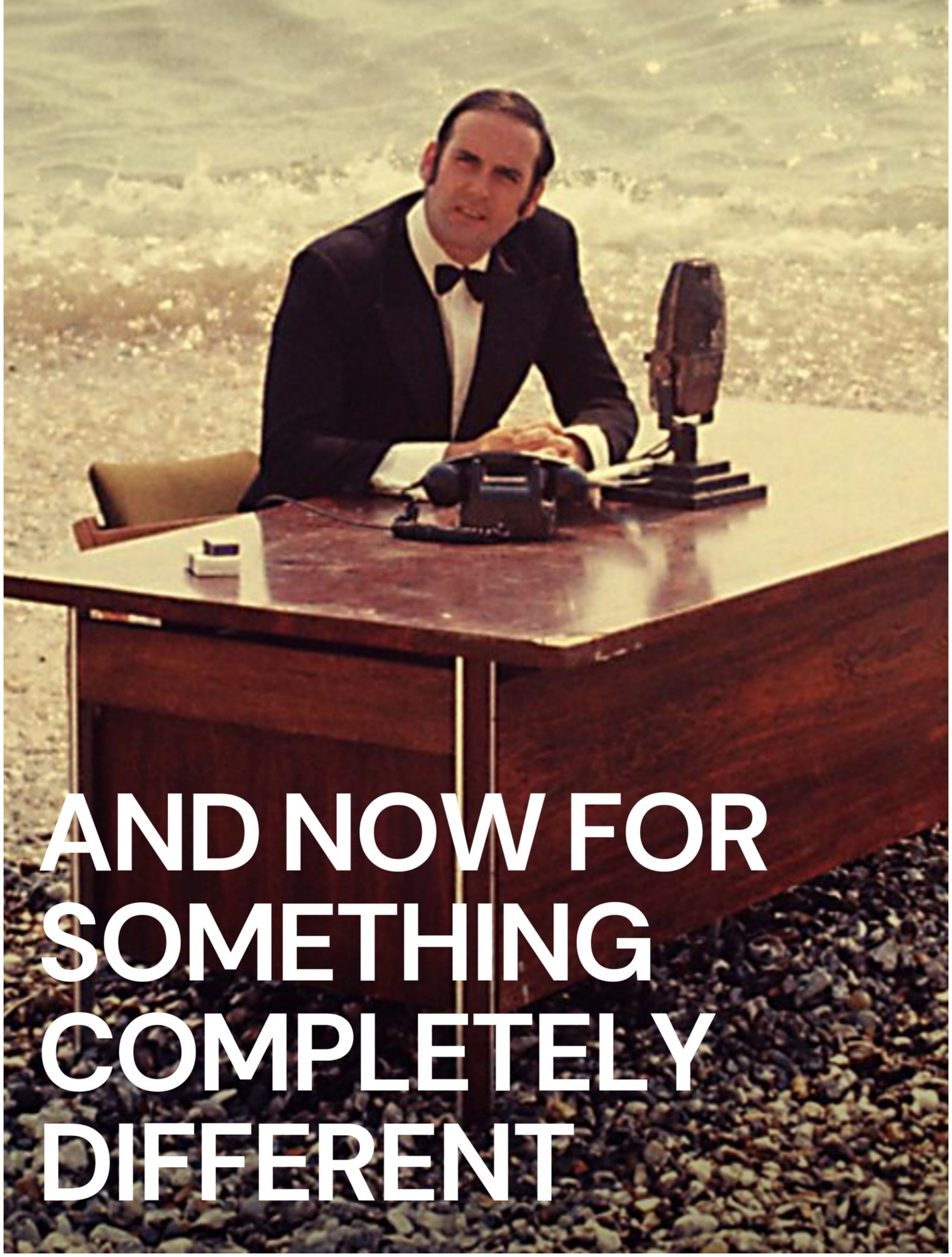
A Monoid Structure on Diagrams

A multiplication formula:

i) Put the first diagram on top of the second, identifying the vertices in the middle

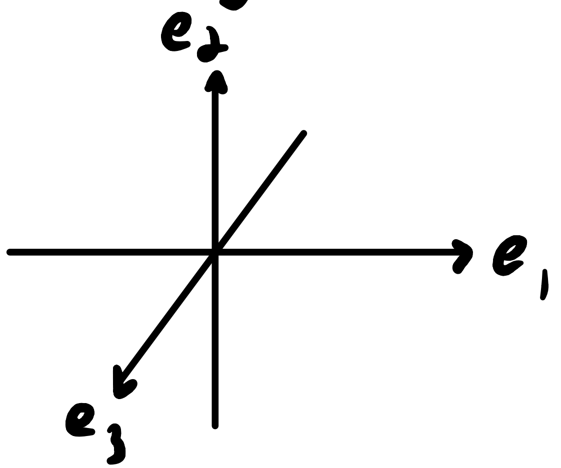
ii) Restrict to the top and bottom, preserving which vertices are connected in the larger diagram.




AND NOW FOR
SOMETHING
COMPLETELY
DIFFERENT

Representations

Ex) $S_3 \circlearrowright \mathbb{R}^3$ by $\sigma . e_i = e_{\sigma(i)}$

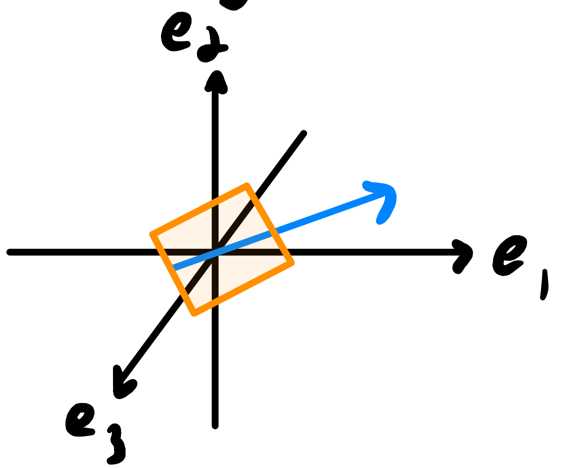


$$(12) \longleftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(123) \longleftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Representations

Ex $S_3 \curvearrowright \mathbb{R}^3$ by $\sigma . e_i = e_{\sigma(i)}$



$V = \{(a,a,a) : a \in \mathbb{R}\}$

$W = \{(a,b,c) : a+b+c=0\}$

$$(12) \longleftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(123) \longleftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Goal: split representations into these smaller, irreducible subrepresentations.

Centralizer Algebras

$\mathrm{End}_G(V)$ = Linear transformations $V \to V$ that commute with the G-action.

Called the centralizer algebra of G acting on V. Think of it like symmetries of symmetries.

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Exercise To describe $\mathrm{End}_{S_3}(\mathbb{R}^3)$, write down the 3×3 matrices that commute with each of the six 3×3 permutation matrices.

Diagram Algebras

Schur-Weyl Duality

V_n : an n-dimensional $\mathbb{C}$-vector space

GL_n : group of $n \times n$ invertible matrices over $\mathbb{C}$

$V_n^{\otimes r}$: the r^{th} tensor power of V_n. Think of elements as sequences

$$v_1 \otimes v_2 \otimes \cdots \otimes v_r$$

with each $v_i \in V_n$ (actually linear combinations of these)

GL_n acts on $V_n^{\otimes r}$ in the following way

$$A \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_r) = (Av_1) \otimes (Av_2) \otimes \cdots \otimes (Av_r)$$

Schur-Weyl Duality

S_r also acts on $V_n^{\otimes r}$ by permuting tensor factors

$$\sigma . (v_1 \otimes v_2 \otimes \cdots \otimes v_r) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}$$

$$GL_n \circlearrowright V_n^{\otimes r} \circlearrowleft S_r$$

Natural question: How do these actions interact with each other?

Schur-Weyl Duality

$$GL_n \circlearrowright V_n^{\otimes r} \circlearrowleft S_r$$

They are mutual centralizers

- $\mathrm{End}_{S_r}(V_n^{\otimes r})$ is generated by the GL_n-action

  Maps $V_n^{\otimes r} \to V_n^{\otimes r}$ which commute with the S_r-action

- $\mathrm{End}_{GL_n}(V_n^{\otimes r})$ is generated by the S_r-action

Schur-Weyl Duality

This is an example of Schur-Weyl duality, first discovered by Schur and then popularized by Weyl who used it to classify U_n and GL_n representations.

Main Takeaway:

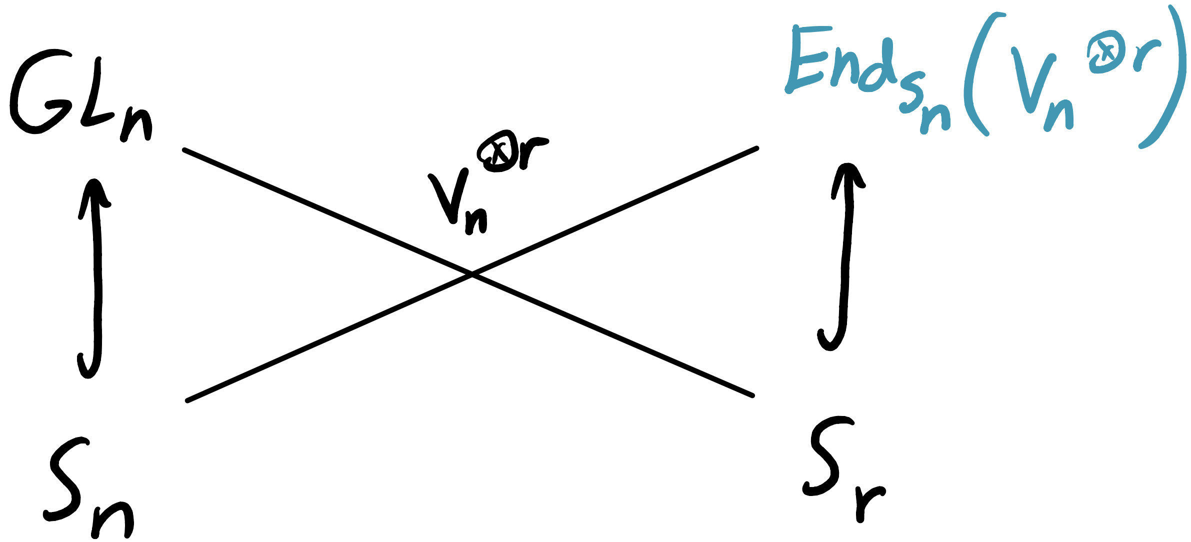
This duality connects the representation theory of the two objects, pairing up their irreducible representations.

More precisely:

$$V_n^{\otimes r} \cong \bigoplus_{\lambda} E^{\lambda} \otimes S^{\lambda} \quad \text{as a } GL_n \times S_r\text{-module}$$

The Partition Algebra

We can restrict the GL_n action to the $n \times n$ permutation matrices



To get a sense for working with these centralizers, lets walk through this classical case.

The Partition Algebra

If V_n has basis $e_1, \ldots, e_n$, then $V_n^{\otimes r}$ has a basis $e_{\underline{i}} = e_{\underline{i}_1} \otimes \cdots \otimes e_{\underline{i}_r}$ indexed by sequences $\underline{i}$ of r elements in $\{1, \ldots, n\}$, so,

$$\dim(V_n^{\otimes r}) = n^r$$

The Partition Algebra

If V_n has basis $e_1, \dots, e_n$, then $V_n^{\otimes r}$ has a basis $e_{\underline{i}} = e_{\underline{i}_1} \otimes \cdots \otimes e_{\underline{i}_r}$ indexed by sequences $\underline{i}$ of r elements in $\{1, \dots, n\}$. so,

$$\dim(V_n^{\otimes r}) = n^r$$

Exercise To describe $\mathrm{End}_{S_{10}}(V_{10}^{\otimes 5})$, compute all the $100{,}000 \times 100{,}000$ matrices that commute with the $10! = 3{,}628{,}800$ permutations in S_{10}.

The Partition Algebra

Or instead, notice that if

$$M = (m_{\underline{i},\underline{j}}) \in \mathrm{End}(V_n^{\otimes r})$$

then

$$M \in \mathrm{End}_{S_n}(V_n^{\otimes r}) \iff m_{\underline{i}\underline{j}} = m_{\sigma(\underline{i}),\sigma(\underline{j})} \quad \forall\, \underline{i}, \underline{v}, \sigma$$

when $\sigma(\underline{i_1}, \dots, \underline{i_r}) = \sigma(\underline{i_1}) \cdots \sigma(\underline{i_r})$

The Partition Algebra

Visualizing matrices in $\mathrm{End}_{S_3}(V_3^{\otimes 2})$:

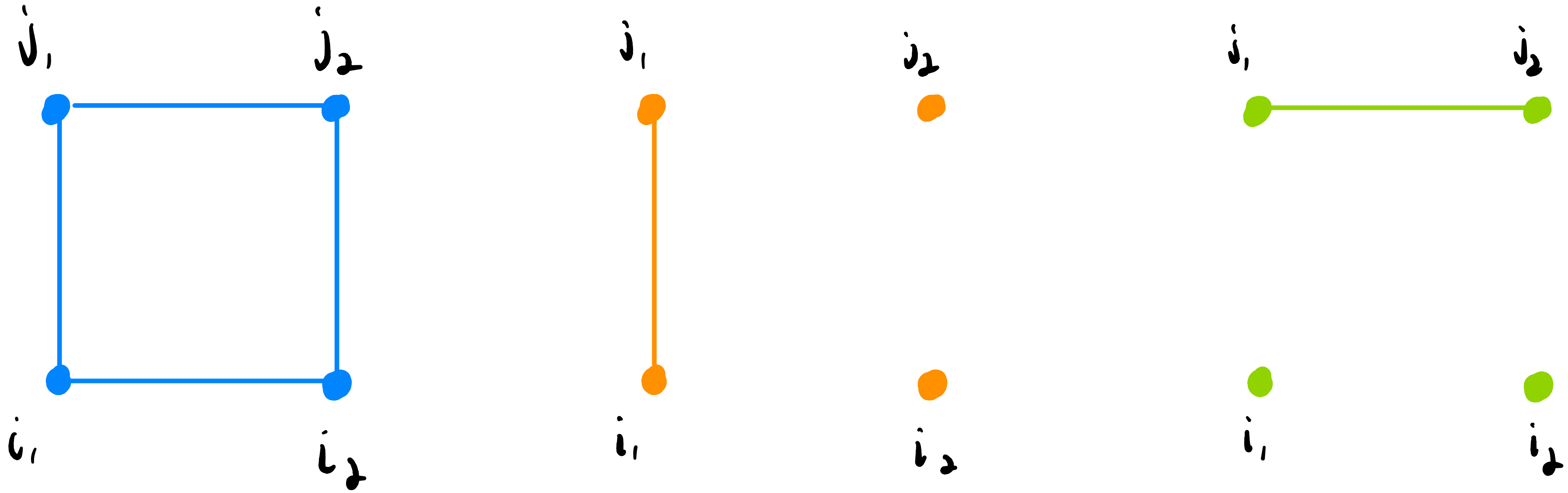
$$
\begin{array}{c|c}
 & \underline{j} \\
\underline{i} & \begin{array}{ccccccccc} 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \end{array} \\
\begin{array}{c} 11 \\ 12 \\ 13 \\ 21 \\ 22 \\ 23 \\ 31 \\ 32 \\ 33 \end{array} &
\begin{bmatrix}
a & & & & & & & & \\
 & & b & & & & & & c \\
 & b & & & c & & & & \\
 & & & & & b & & & c \\
 & & & & a & & & & \\
c & & & b & & & & & \\
 & & & & c & & & b & \\
c & & & & & & b & & \\
 & & & & & & & & a
\end{bmatrix}
\end{array}
$$

$\underline{\sigma}$			
ε	$\underline{(11,11)}$	$\underline{(12,13)}$	$\underline{(12,33)}$
(12)	22, 22	21, 23	21, 33
(13)	33, 33	32, 31	32, 11
(23)	11, 11	13, 12	13, 22
(123)	22, 22	23, 21	23, 11
(132)	33, 33	31, 32	31, 22

Each orbit represents a basis element, so how do we compactly represent each orbit?

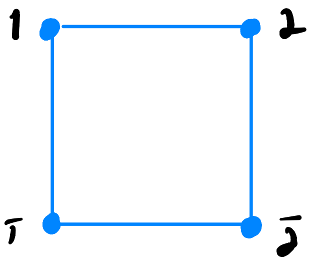
The Partition Algebra

σ				
ε	(11, 11)	(12, 13)	(12, 33)	$(i_1 i_2, j_1 j_2)$
(12)	22, 22	21, 23	21, 33	
(13)	33, 33	32, 31	32, 11	
(23)	11, 11	13, 12	13, 22	
(123)	22, 22	23, 21	23, 11	
(132)	33, 33	31, 32	31, 22	

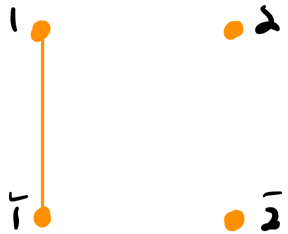


The Partition Algebra

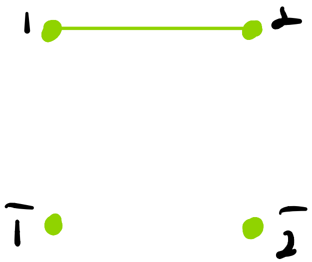
If we label these graphs with $1, \ldots, r$ on top and $\bar{1}, \ldots, \bar{r}$ on bottom, we get set partitions from connected components.



$\longrightarrow \{\{1, 2, \bar{1}, \bar{2}\}\}$



$\longrightarrow \{\{1, \bar{1}\}, \{2\}, \{\bar{2}\}\}$

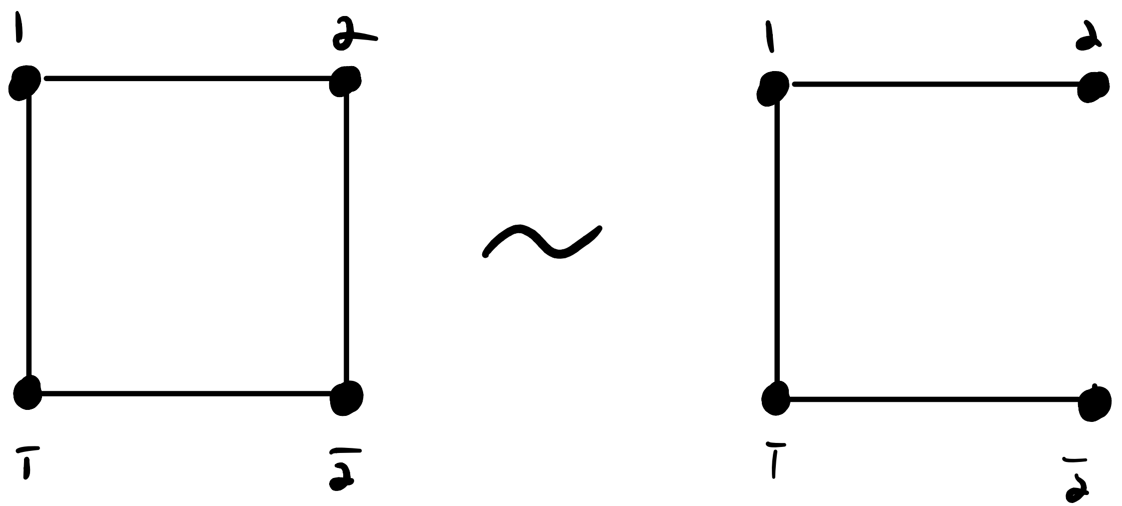


$\longrightarrow \{\{1, 2\}, \{\bar{1}\}, \{\bar{2}\}\}$

Write Π_{2r} for the set of set partitions of $[r] \cup [\bar{r}]$.

The Partition Algebra

These graphs representing orbits are not unique:

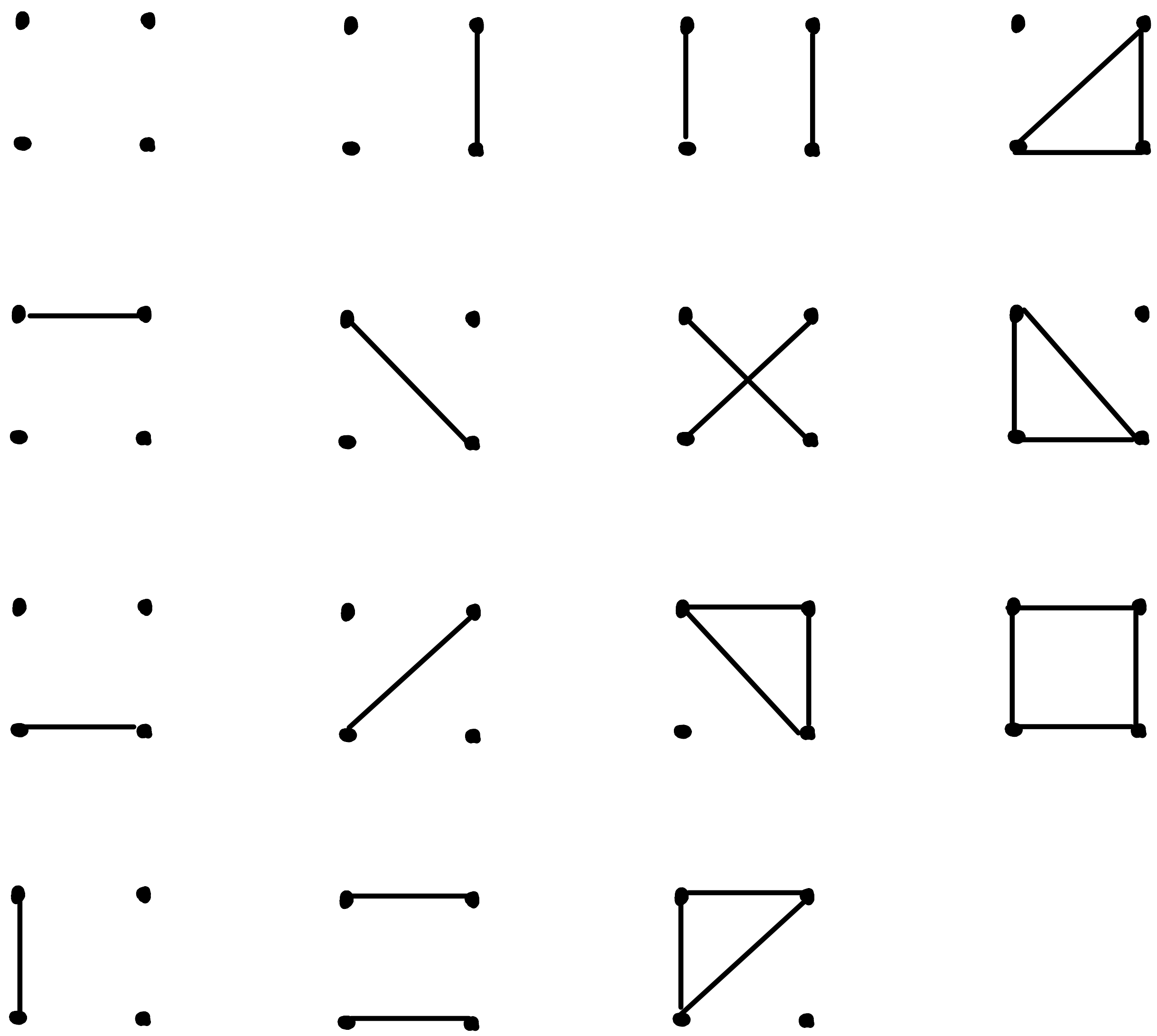


A diagram is an equivalence class of graphs on the vertices $[r] \cup [\bar{r}]$ with the same connected components

They are in correspondence with set partitions in Π_{2r}

The Partition Algebra

For example, $\mathrm{End}_{S_4}(V_4^{\otimes 2})$ has a basis indexed by:



(need $n \geq 2r$ for all the diagrams to appear)

The Partition Algebra

We'll now call $\mathrm{End}_{S_n}(V_n^{\otimes r})$ the Partition algebra $P_r(n)$ (introduced by Jones and by P. Martin in the 90s)

The basis obtained this way is called the orbit basis, which we'll write as

$$\{ T_\pi \; : \; \pi \in \Pi_{2r} \}$$

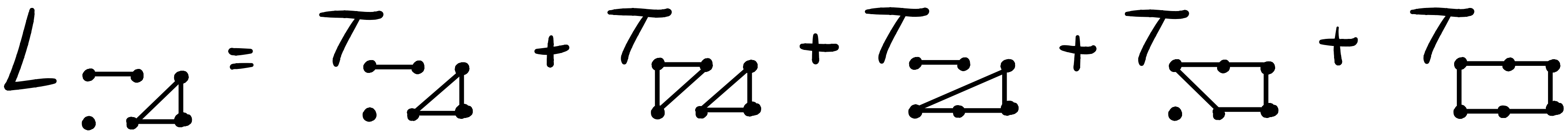
The Partition Algebra

There is another basis $\{L_\pi\}$ called the diagram basis given by:

$$L_\pi = \sum_{\nu \leq \pi} T_\nu$$

ν is a coarsening of π

Ex)



The Partition Algebra

Orbit basis example:

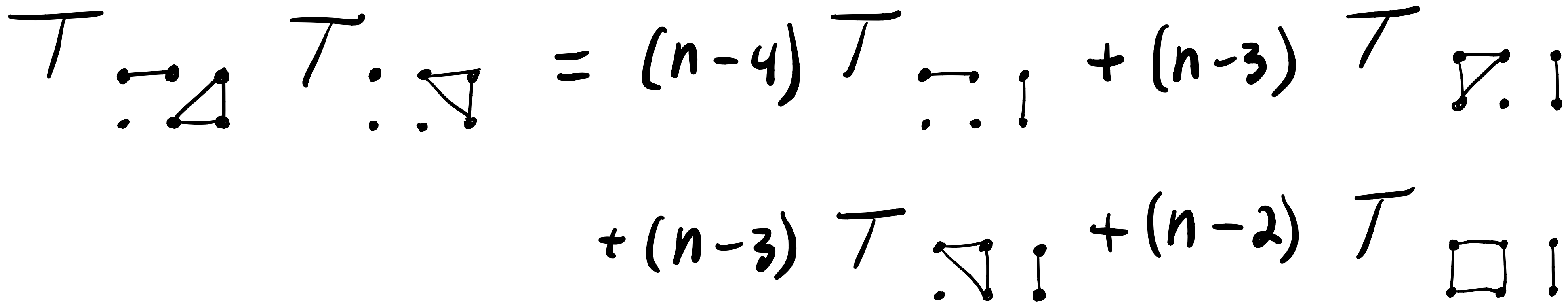
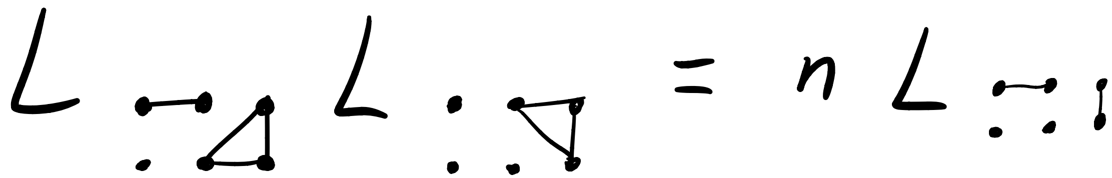


Diagram basis example:



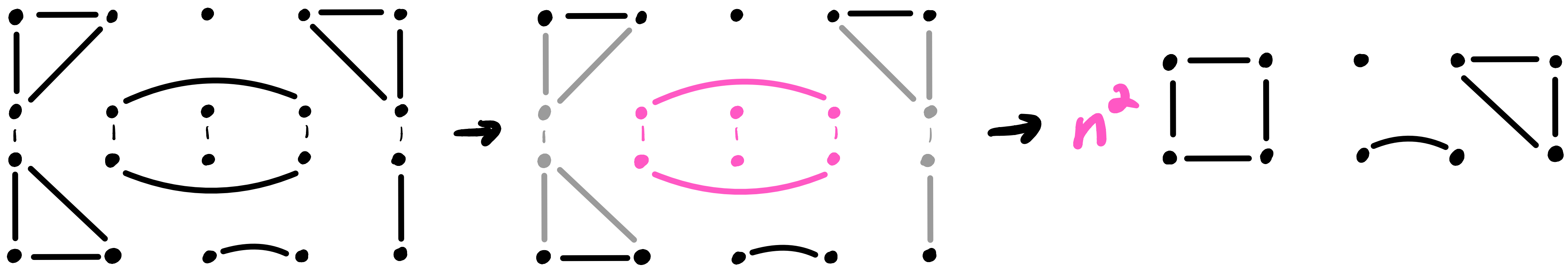
The Partition Algebra

The formula:

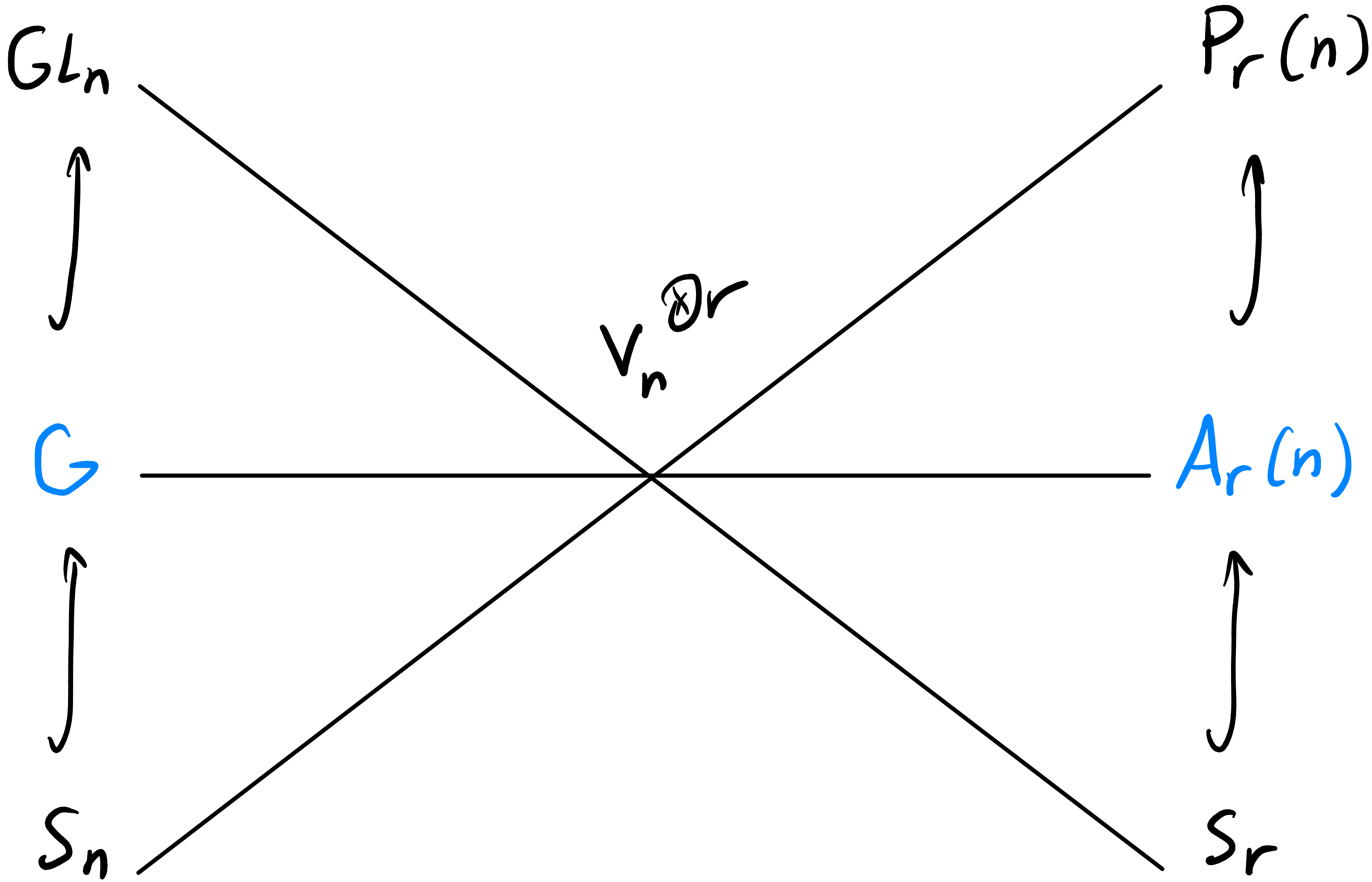
i) Put the first diagram on top of the second

ii) Restrict to the top and bottom, preserving which vertices are connected in the larger diagram.

iii) Record a coefficient of n^c where c is the number of components stranded in the middle.

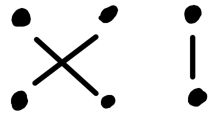
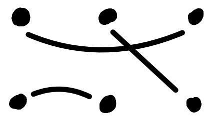
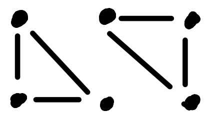


The Partition Algebra



The Partition Algebra

$$G \circlearrowright V_n^{\otimes r} \circlearrowleft A_r(n)$$

$\underline{G}$	$\underline{A_r(n)}$	Typical Element	
GL_n	$\mathbb{C}S_r$		
O_n	Brauer Algebra ($B_r(n)$)		(matchings)
S_n	Partition Algebra		

Recap

- Representation theory of $P_r(n)$ and S_n are connected.

- $P_r(n)$ comes with a natural orbit basis.

- $P_r(n)$ and its subalgebras have a beautiful diagrammatic product when viewed in the right basis.

Painted Algebras

Howe Duality

$V_{n,k}$: The space of $n \times k$ matrices over $\mathbb{C}$

$P^r(V_{n,k})$: The space of homogeneous polynomial forms on $V_{n,k}$

These are homogeneous polynomials of degree r in indeterminates

$$x_{ij} \quad \text{for} \quad 1 \leq i \leq n, \quad 1 \leq j \leq k$$

where x_{ij} picks out the entry ij in the matrix:

$$x_{12}\, x_{13}\, x_{22}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right) = 2 \cdot 3 \cdot 5$$

Howe Duality

In the 1980s, Roger Howe determined that

$$GL_n \circlearrowright \mathcal{P}^r(V_{n,k}) \circlearrowleft GL_k$$

form a mutually centralizing pair where

- $A \in GL_n$ acts by $(A.f)(X) = f(A^{-1}X)$
- $B \in GL_k$ acts by $(B.f)(X) = f(XB)$

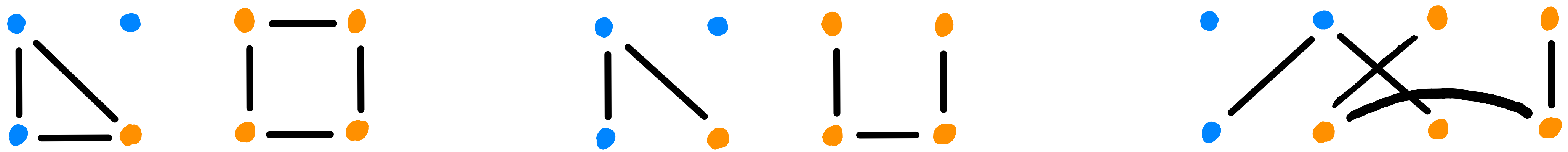
Howe Duality

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
GL_n \arrow[drr, dash] & P^r(V_{n,k}) & \mathrm{End}_{S_n}(P^r(V_{n,k})) \arrow[dll, dash] \\
S_n \arrow[u, hook] & & GL_k \arrow[u, hook]
\end{tikzcd}
$$

The Multiset Partition Algebra

Orellana and Zabrocki (2020) examined $\mathrm{End}_{S_n}(P^r(V_{n,\kappa}))$, describing an orbit basis for it and naming it $MP_{r,\kappa}(n)$, the multiset partition algebra.

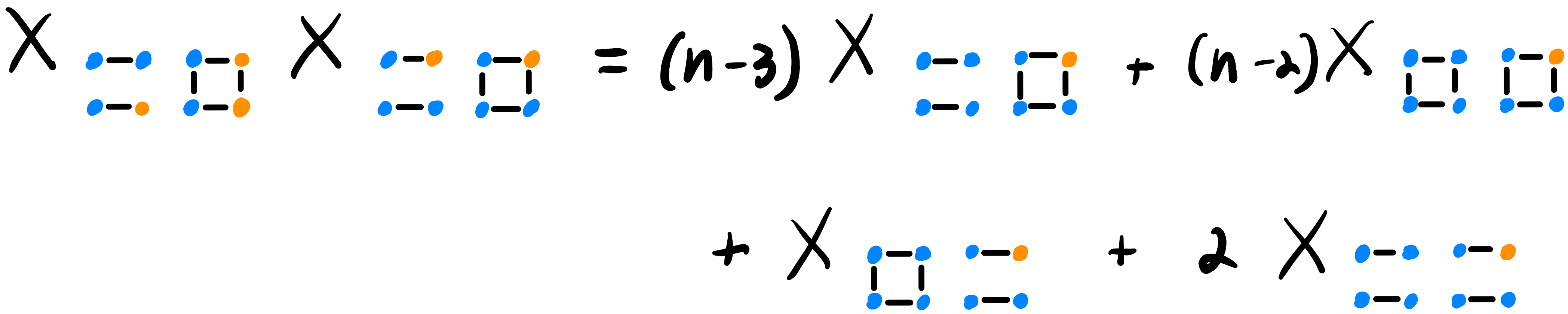
This basis is indexed by diagrams whose vertices are colored from a set of κ colors with identically colored vertices among the top or bottom indistinguishable.



Write $\tilde{\Pi}_{2r,\kappa}$ for the set of these diagrams.

The Multiset Partition Algebra

Writing $\{X_{\tilde{\pi}} : \tilde{\pi} \in \tilde{\Pi}_{2r,k}\}$ for the orbit basis obtained by Orellana and Zabrocki, an example of its multiplication is:



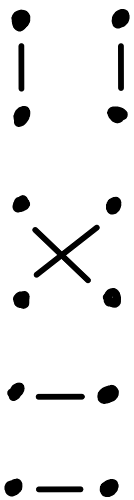
This looks like the orbit basis for $P_r(n)$. Can we change to a basis like the diagram basis?

The Multiset Partition Algebra

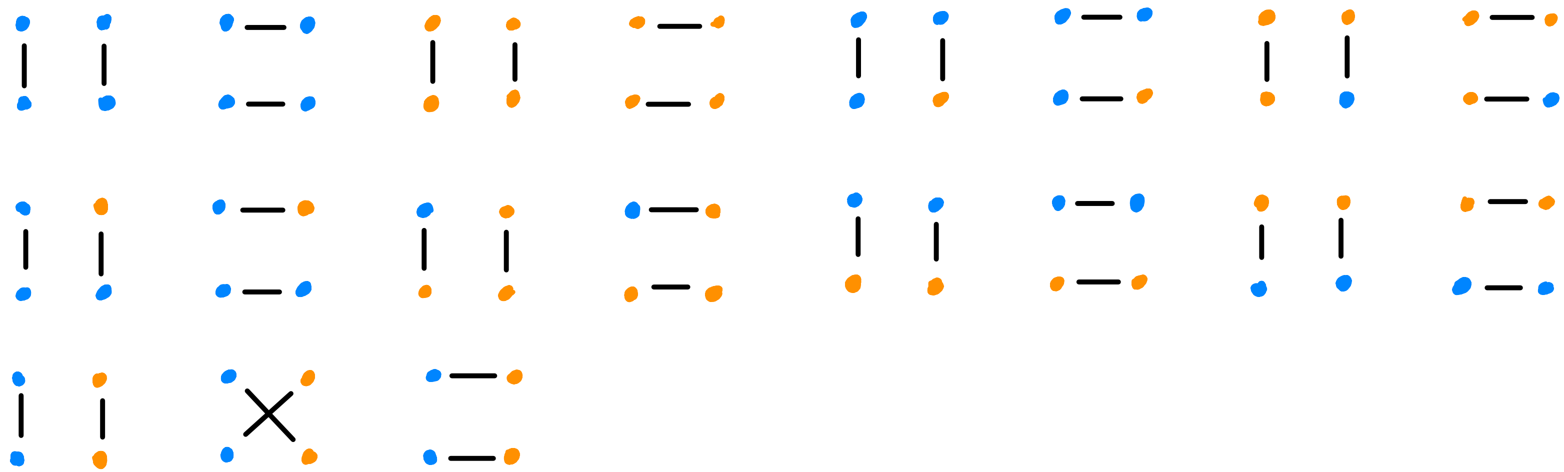
Let $S_r \subseteq A_r(n) \subseteq P_r(n)$ and define a new algebra $\tilde{A}_{r,k}(n)$ called the corresponding Painted algebra with basis:

$$\left\{ D_{\tilde{\pi}} : \tilde{\pi} \text{ obtained by coloring the vertices of a diagram in } A_r(n) \right\}$$

$B_2(n)$

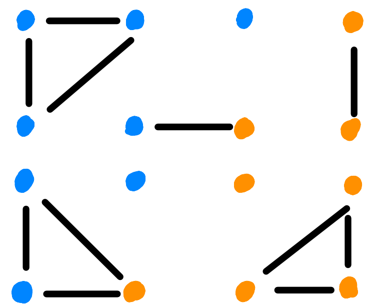


$\tilde{B}_{2,2}(n)$



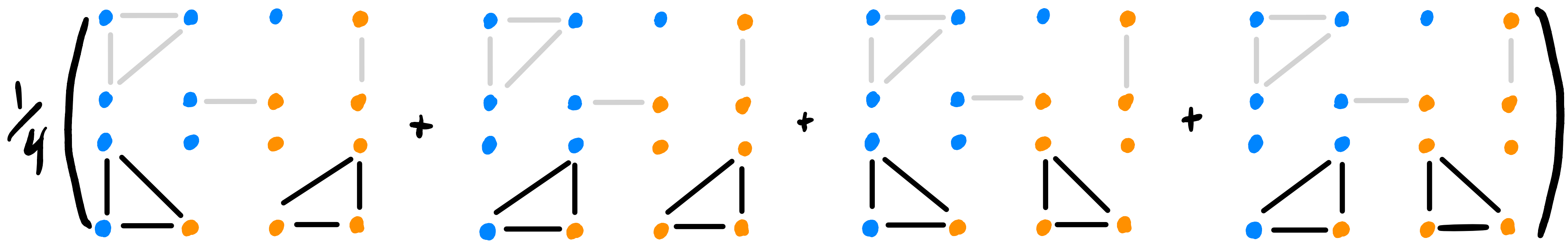
The Multiset Partition Algebra

The product is given by:

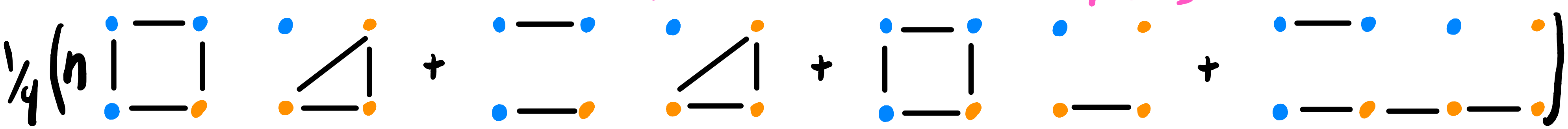


Colors must match in middle or else product is zero

Average over permutations of the top of the second diagram



Take the product as in $P_r(n)$



The Multiset Partition Algebra

Theorem (W'23) Let $S_n \subseteq G \subseteq GL_n$ be a subgroup with

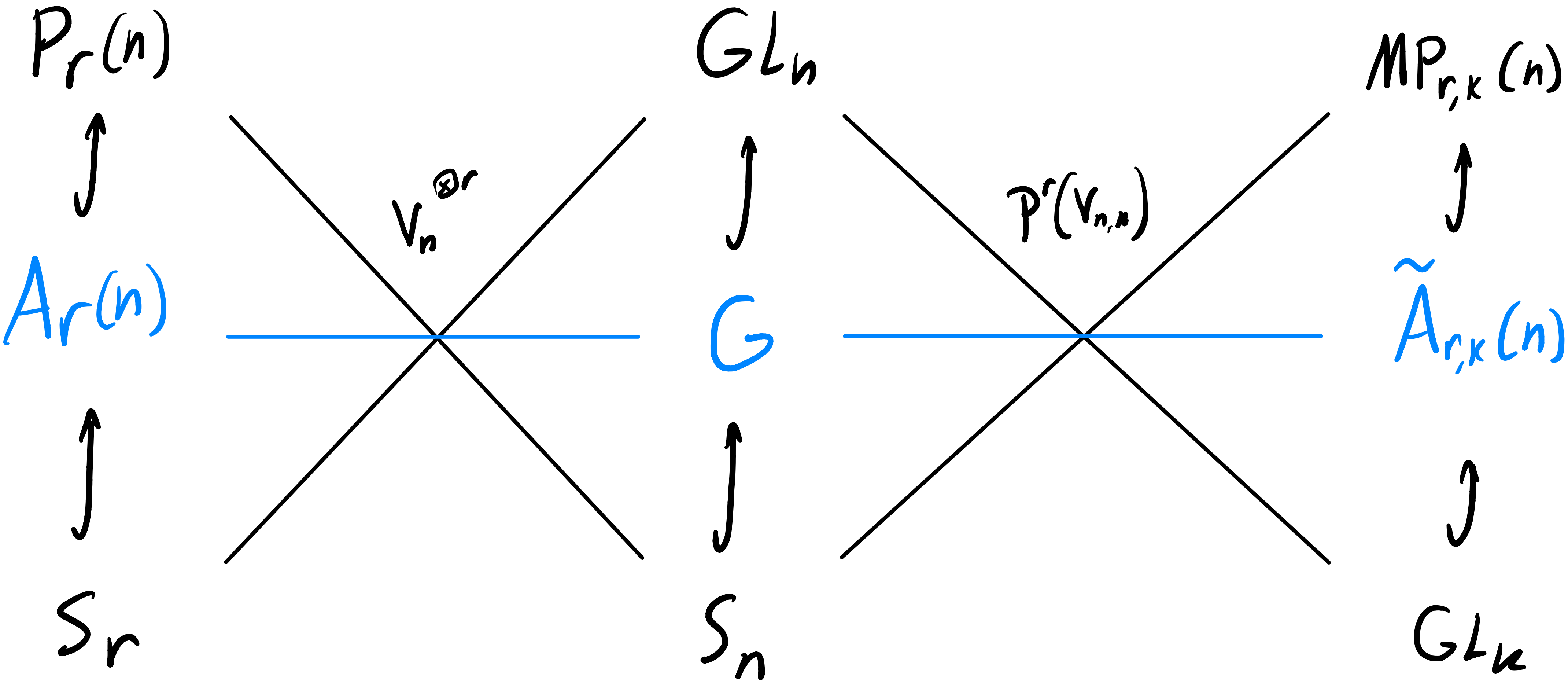
$$\mathrm{End}_G(V_n^{\otimes r}) = A_r(n).$$

Then

$$\mathrm{End}_G(P^r(V_{n,k})) \cong \tilde{A}_{r,k}(n).$$

Corollary (W'23) $MP_{r,k}(n) \cong \tilde{P}_{r,k}(n)$. We call the basis $\{D_{\tilde{\pi}}\}$ of $MP_{r,k}(n)$ the diagram-like basis

Subalgebras



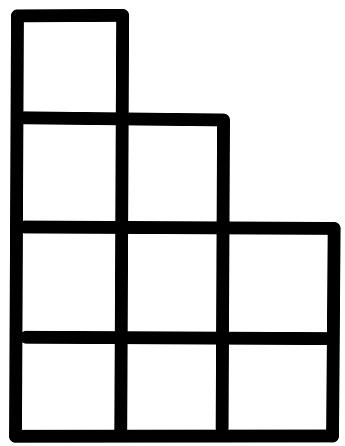
Representations

Representations

An integer partition is a weakly decreasing sequence $(\lambda_1, \ldots, \lambda_\ell)$ of positive integers.

We write $\lambda \vdash n$ to mean $\lambda_1 + \cdots + \lambda_\ell = n$.

The Young diagram of λ is an array of left-justified boxes with λ_i boxes in the i^{th} row from the bottom.

$(3, 3, 2, 1) \longleftrightarrow$ 

Representations

A **standard Young tableau** of shape $\lambda \vdash n$ is a filling of λ's Young diagram with $1, \ldots, n$ so that the rows and columns are increasing.

Write S^λ for the $\mathbb{C}$-span of SYT of shape λ

$$S^{(3)} = \mathbb{C}\left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \right\}$$

$$S^{(2,1)} = \mathbb{C}\left\{ \begin{array}{|c|c}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|c}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \right\}$$

Representations

For $\lambda \vdash n$, S^λ is a representation of S_n:

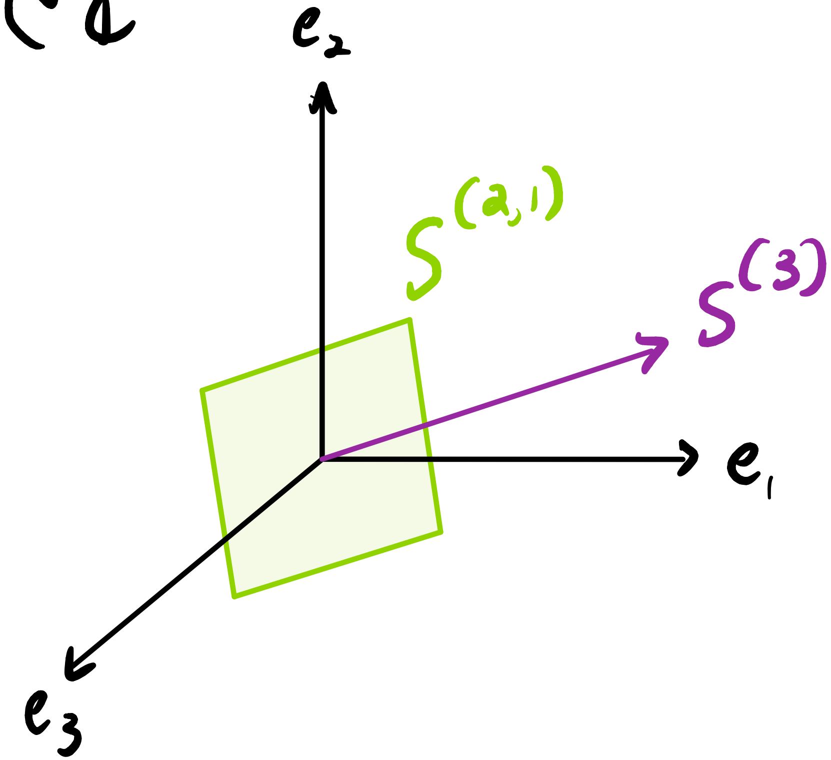
$$(1\,3\,2) \cdot \begin{array}{|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array} = \begin{array}{|c|c|} \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 3 & 1 \\ \hline \end{array} = \begin{array}{|c|c|} \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} - \begin{array}{|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array}$$

"straightening algorithm"

Each S^λ is irreducible and every irreducible representation is isomorphic to some S^λ.

Representations

$S_3 \circlearrowright \mathbb{C}^3$



$$S^{(3)} = \mathbb{C}\left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \right\}$$

$$S^{(2,1)} = \mathbb{C}\left\{ \begin{array}{|c|c}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\,,\ \begin{array}{|c|c}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \right\}$$

$$\mathbb{C}^3 \cong S^{(3)} \oplus S^{(2,1)}$$

Representations

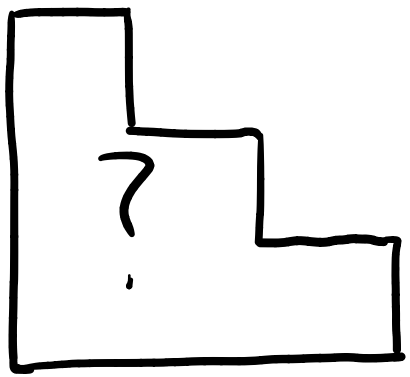
By the duality of the actions,

$$\mathcal{P}^r(V_{n,k}) \cong \bigoplus_{\lambda} S^\lambda \otimes MP^\lambda_{r,k}$$

This pairs up irreducible representations

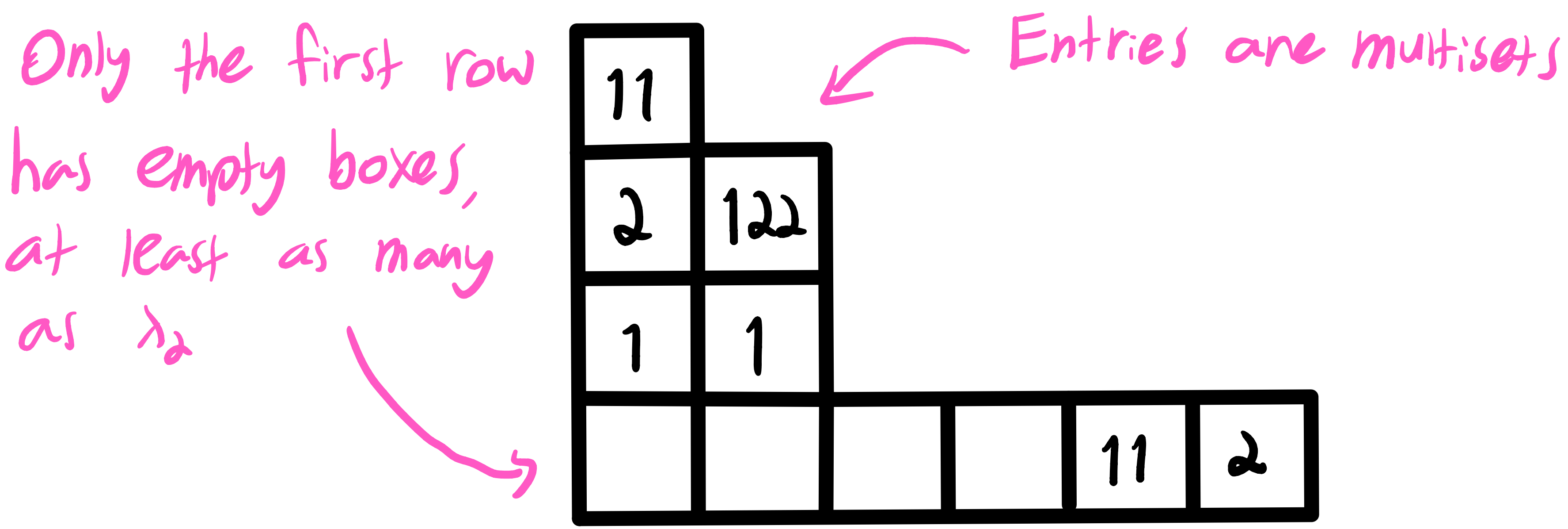
$$S^\lambda \longleftrightarrow MP^\lambda_{r,k}$$

5		
2	4	
1	3	6



Representations

A multiset partition tableau of shape λ is a filling of λ's Young diagram like so:

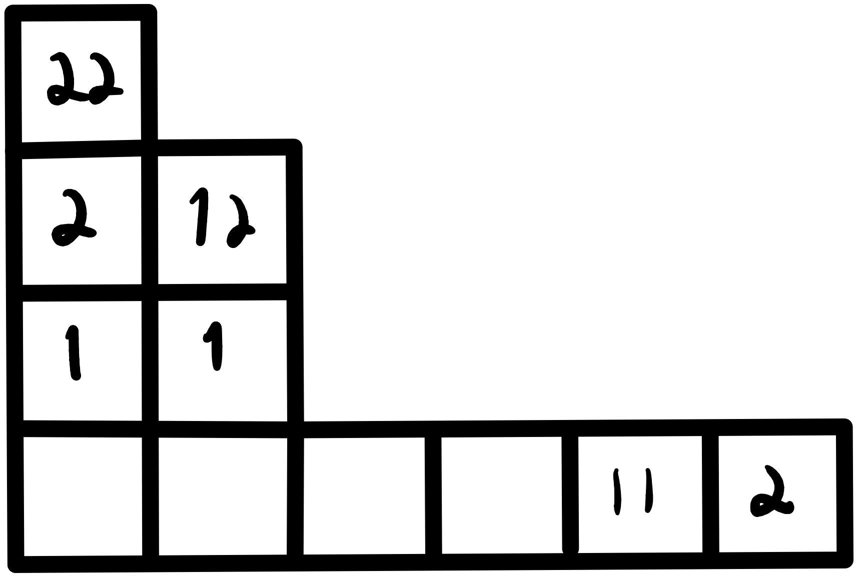


Representations

Order multisets by the last-letter order:

$11 < 2 \qquad 12 < 22 \qquad 22 < 122$

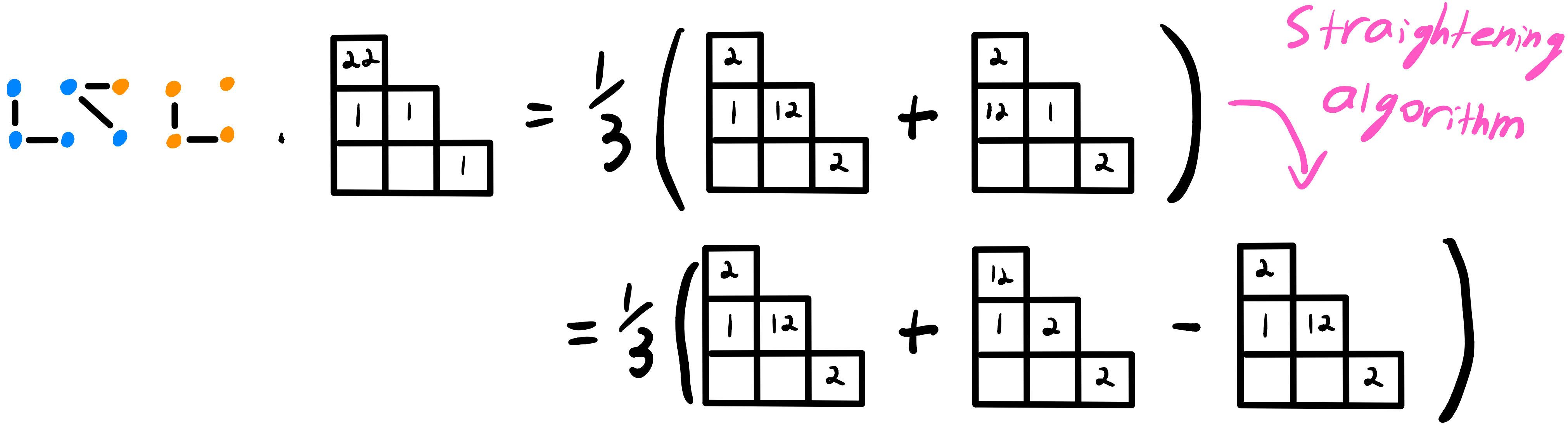
A semistandard multiset partition tableau has rows weakly increasing and columns strictly increasing.



Write $MP^{\lambda}_{r,n}$ for the $\mathbb{C}$-span of these with r numbers from $1, \ldots, k$.

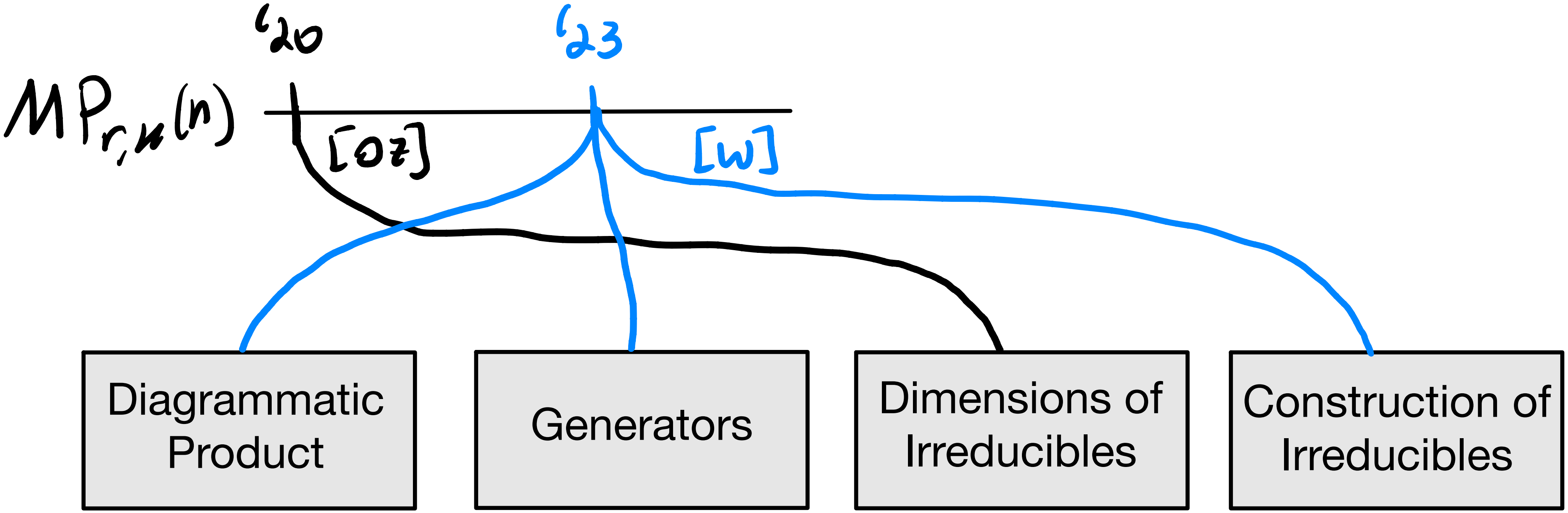
Representations

There is an action of $MP_{r,k}(n)$ on $MP^{\lambda}_{r,k}$

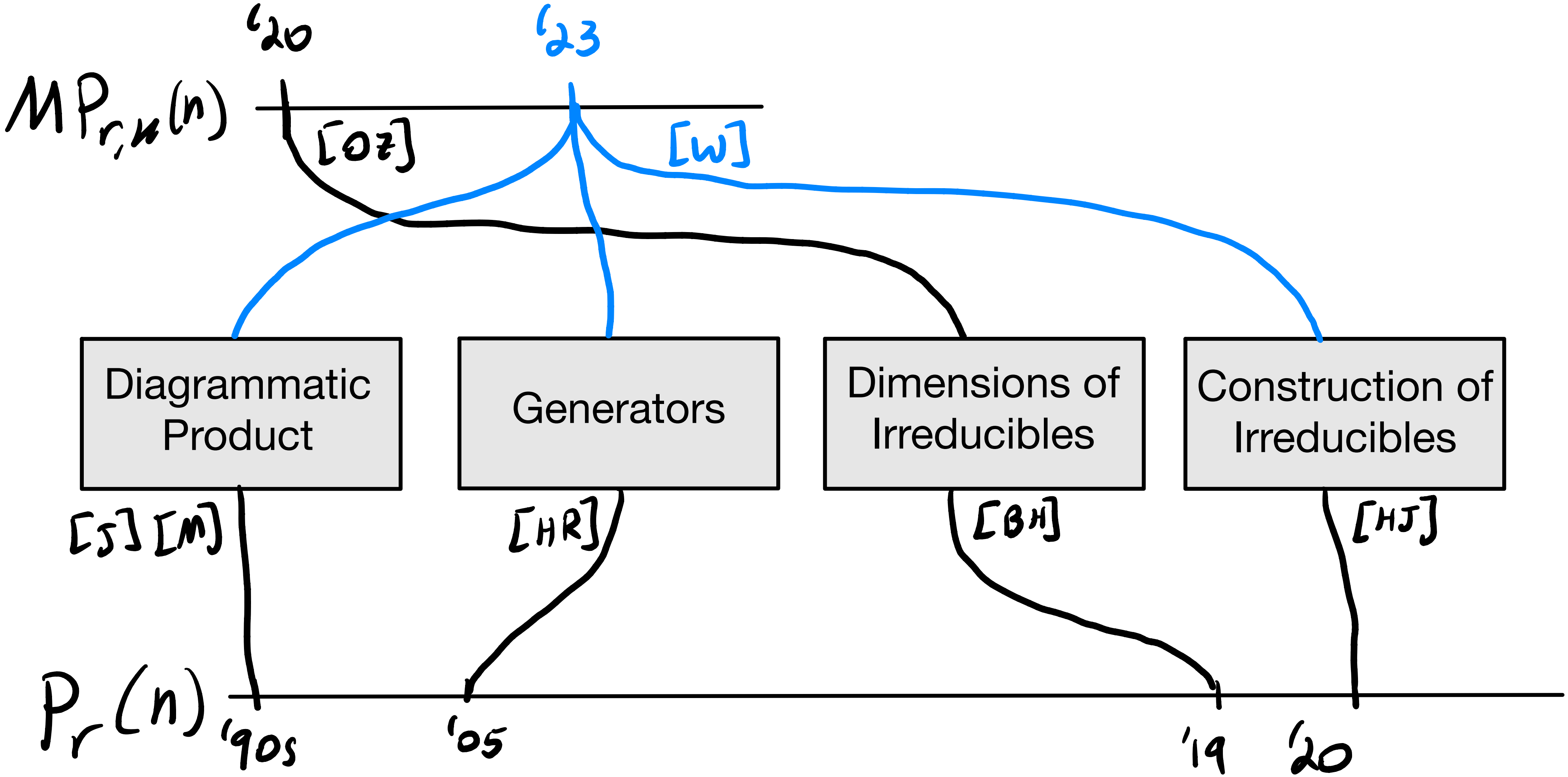


Theorem(w) The $MP^{\lambda}_{r,k}$ for $\lambda \vdash n$ and $\sum_{i=2}^{\ell(\lambda)} \left\lceil \frac{i-1}{k} \right\rceil \lambda_i \leq r$ form a complete set of irreducible representations for $MP_{r,k}(n)$ when $n \geq 2r$.

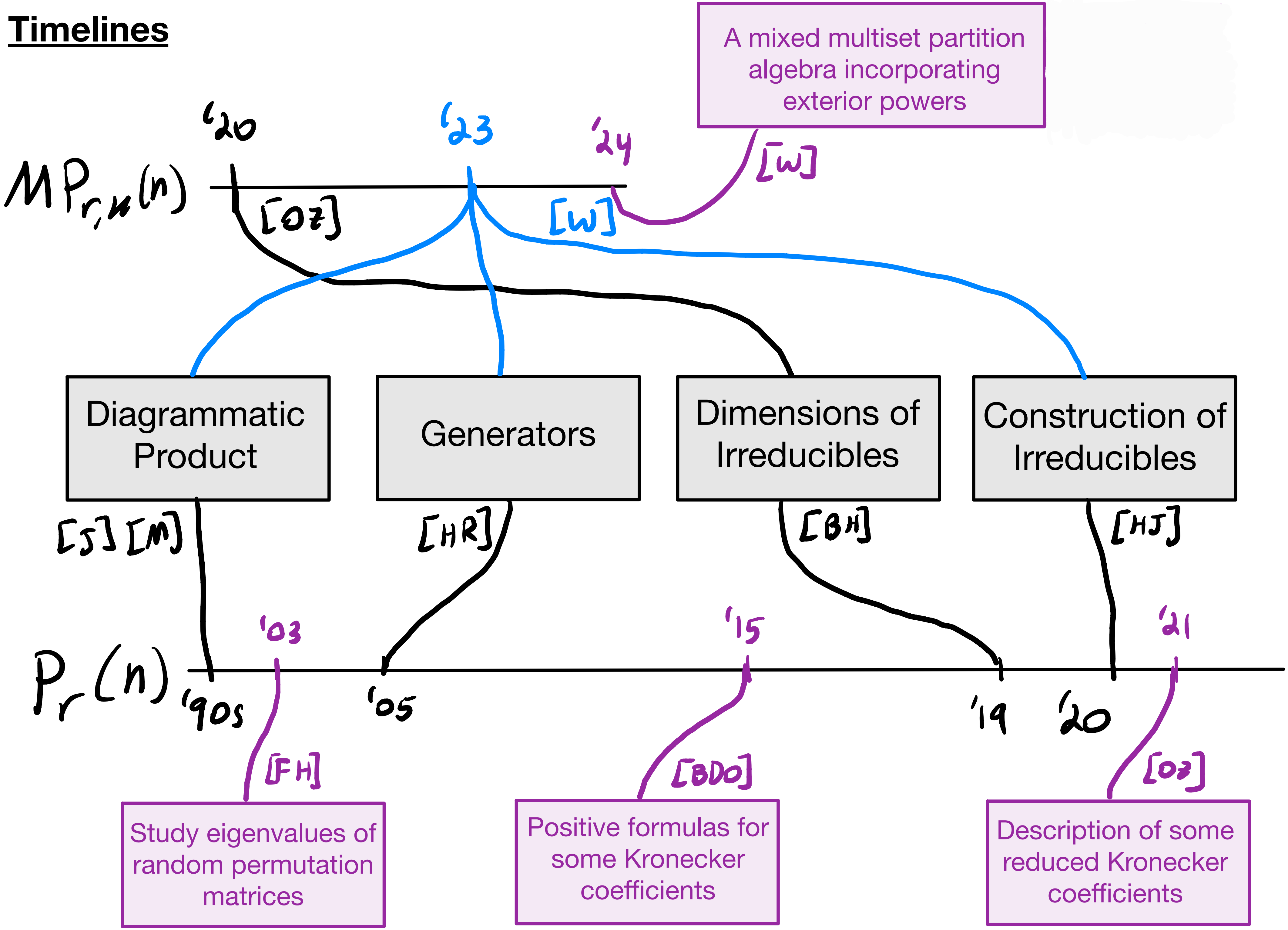
Timelines


'20
'23
$MP_{r,k}(n)$
[OZ]
[W]
Diagrammatic Product
Generators
Dimensions of Irreducibles
Construction of Irreducibles

Timelines


'20
'23
$MP_{r,k}(n)$
[OZ]
[W]
Diagrammatic Product
Generators
Dimensions of Irreducibles
Construction of Irreducibles
[J] [M]
[HR]
[BH]
[HJ]
$P_r(n)$
'90s
'05
'19
'20

Timelines

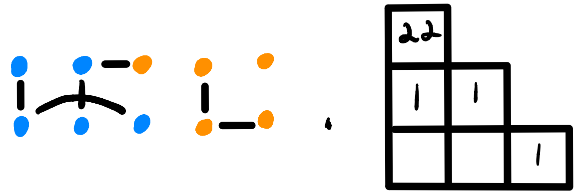


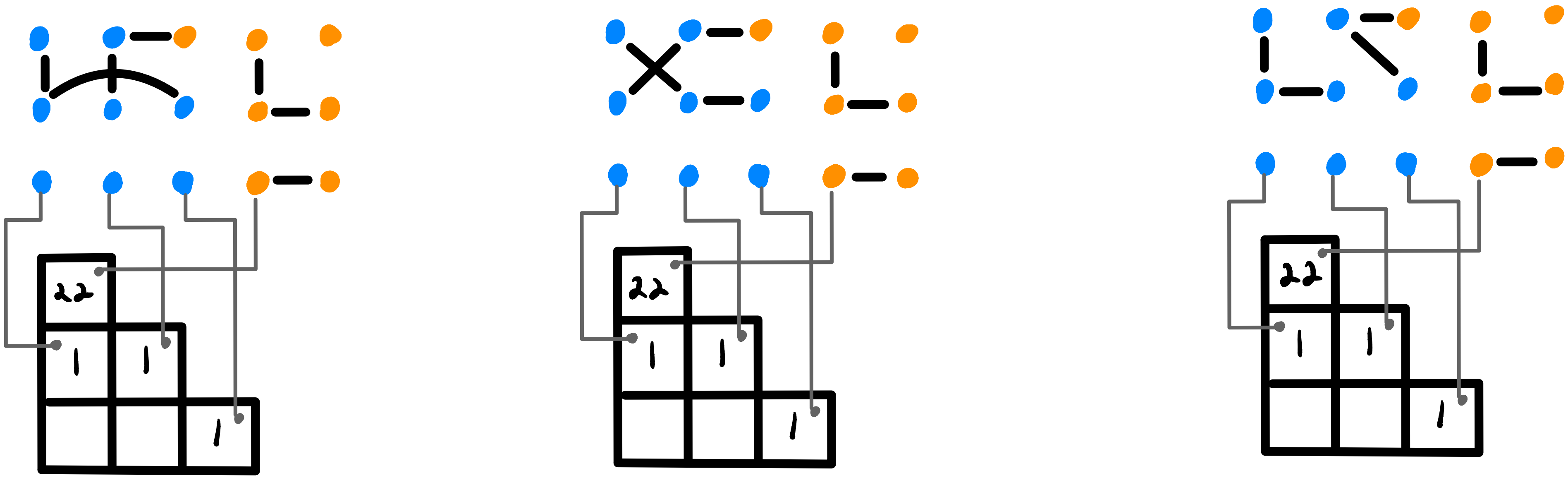
A mixed multiset partition algebra incorporating exterior powers
'20
'23
'24
[W]
$MP_{r,k}(n)$
[OZ]
[W]
Diagrammatic Product
Generators
Dimensions of Irreducibles
Construction of Irreducibles
[S] [M]
[HR]
[BH]
[HJ]
'03
'15
'21
$P_r(n)$
'90s
'05
'19
'20
[FH]
[BDO]
[OZ]
Study eigenvalues of random permutation matrices
Positive formulas for some Kronecker coefficients
Description of some reduced Kronecker coefficients

Thank you!

Representations

An example of the action:





* Two blocks above the first row get combined

Proof Sketch

Break $P^r(V_{n,k})$ into pieces $\mathcal{U}_{\underline{a}}$ based on the second index.

E.g. $x_{11}x_{21}x_{22}x_{22} \in \mathcal{U}_{(2,2)}$

Write $W_{r,k}$ for weak compositions of r of length k

Then $P^r(V_{n,k}) \cong \bigoplus_{\underline{a} \in W_{r,k}} \mathcal{U}_{\underline{a}}$ as a GL_n-module

Proof Sketch

$S_{\underline{a}}$: Young subgroup $\qquad s_{\underline{a}} = \frac{1}{|S_{\underline{a}}|} \sum_{\sigma \in S_{\underline{a}}} \sigma$

E.g. $\quad S_{(2,2)} = S_{\{1,2\}} \times S_{\{3,4\}}$

$$s_{(2,2)} = \frac{1}{4}\left(1234 + 2134 + 1243 + 2143\right)$$

Recall S_r acts on $V_n^{\otimes r}$ by permuting factors

$$s_{(2,2)}(e_1 \otimes e_2 \otimes e_2 \otimes e_2) = \frac{1}{2}\left(e_1 \otimes e_2 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 \otimes e_2\right)$$

Proof Sketch

As vector spaces,

$$\Phi : \mathcal{U}_{\underline{a}} \xrightarrow{\sim} s_{\underline{a}} V_n^{\otimes r}$$

$$x_{11} x_{21} x_{22} x_{22} \longmapsto s_{(2,2)} (e_1 \otimes e_2 \otimes e_2 \otimes e_2)$$

They both have a GL_n-action but are not clearly isomorphic as GL_n-modules.

For $M \in GL_n$,

$$\Phi M = M^{-1} \Phi$$

Proof Sketch

However, we get an induced isomorphism

$$\mathrm{End}_G\Big(\bigoplus_{\underline{a}\in W_{r,n}} U_{\underline{a}}\Big) \cong \mathrm{End}_G\Big(\bigoplus_{\underline{a}\in W_{r,n}} s_{\underline{a}} V_n^{\otimes r}\Big)$$

$$\psi \longmapsto \underline{\Phi} \circ \psi \circ \underline{\Phi}^{-1}$$

Note for $M \in GL_n$,

$$\underline{\Phi}\,\psi\,\underline{\Phi}^{-1} M = \underline{\Phi}\,\psi M^{-1}\underline{\Phi}^{-1} = \underline{\Phi} M^{-1}\psi\,\underline{\Phi}^{-1} = M\underline{\Phi}\,\psi\,\underline{\Phi}^{-1}$$

Proof Sketch

$$\mathrm{End}_G(\mathcal{P}^r(V_{n,u})) \cong \mathrm{End}_G\left(\bigoplus_{\underline{a}} s_{\underline{a}} V_n^{\otimes r}\right)$$

$$\cong \bigoplus_{\underline{a},\underline{b}} \mathrm{Hom}_G\left(s_{\underline{b}} V_n^{\otimes r}, s_{\underline{a}} V_n^{\otimes r}\right)$$

$$\cong \bigoplus_{\underline{a},\underline{b}} s_{\underline{a}} \, \mathrm{End}_G(V_n^{\otimes r}) \, s_{\underline{b}}$$

with product

$$(s_{\underline{a}} \pi s_{\underline{b}}) \cdot (s_{\underline{c}} \gamma s_{\underline{d}}) = \delta_{\underline{b},\underline{c}} (s_{\underline{a}} \pi s_{\underline{b}} \gamma s_{\underline{d}})$$

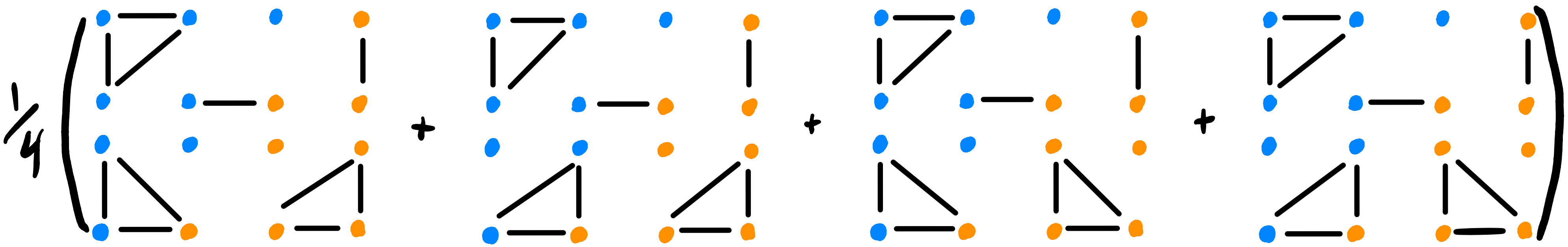
The Multiset Partition Algebra

Colors match in middle

$$(s_{\underline{a}}\, \pi\, s_{\underline{b}}) \cdot (s_{\underline{c}}\, \gamma\, s_{\underline{d}}) = \delta_{\underline{b},\underline{c}} \left(s_{\underline{a}}\, \pi\, s_{\underline{b}}\, \gamma\, s_{\underline{d}}\right)$$

$$= \delta_{\underline{b},\underline{c}} \frac{1}{|S_{\underline{b}}|} \sum_{\sigma \in S_{\underline{b}}} s_{\underline{a}}\, \pi\, \sigma\, \gamma\, s_{\underline{d}}$$

permutations of top of the second diagram



Proof Sketch

Proof summary

- Decompose $P^r(V_{n,k})$
- Leads to a decomposition of $\mathrm{End}_G(P^r(V_{n,k}))$ via idempotents
- The diagram-like basis comes from sandwiching an idempotent between two partition diagrams